

A Geometric View on Inner Transformation between the Variables of a Linear Regression Model

Zhaoyang Li¹, Bostjan Antoncic^{2*}

¹Shanghai University of International Business and Economics, Shanghai, China

²School of Economics and Business, University of Ljubljana, Ljubljana, Slovenia

Email: *leezhya@163.com

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Abstract

In the teaching and researching of linear regression analysis, it is interesting and enlightening to explore how the dependent variable vector can be inner-transformed into regression coefficient estimator vector from a visible geometrical view. As an example, the roadmap of such inner transformation is presented based on a simple multiple linear regression model in this work. By applying the matrix algorithms like singular value decomposition (SVD) and Moore-Penrose generalized matrix inverse, the dependent variable vector lands into the right space of the independent variable matrix and is metamorphosed into regression coefficient estimator vector through the three-step of inner transformation. This work explores the geometrical relationship between the dependent variable vector and regression coefficient estimator vector as well as presents a new approach for vector rotating.

Keywords

Matrix Singular Value Decomposition, Moore-Penrose Generalized Inverse, Matrix Inner Transformation, Regression Analysis

1. Introduction

A matrix can be factorized into the product of several matrices with special properties. Particularly, by singular value decomposition (SVD) which is widely used in regression analysis [1] [2] [3], a matrix can be factorized into the product of three matrices with orthogonal or diagonal properties respectively. SVD can be formularized like $X = U\theta V^T$ [4] [5] [6] [7].

Moore-Penrose generalized inverse is a special case of generalized matrix in-

verse [8] [9] [10] [11] [12] and can be applied in regression analysis [13] [14] and least square analysis [15]. For example, in the study of Tian and Zhang [13], the linear unbiased estimator of partial coefficients is derived through Moore-Penrose generalized inverse algorithm. Herein, multiplied by Moore-Penrose generalized inverse of the independent variable matrix, the dependent variable vector is transformed into the new coordinate systems, or left space and then right space of the independent variable matrix [16]. In fact, this process can also be regarded as a vector rotating algorithm. In addition to Moore-Penrose's generalized inverse algorithm, singular value decomposition (SVD) facilitates such transformation by dividing the transformation process into three-step to present a visible geometrical view.

2. y Can Be Transformed into $\hat{\beta}$

In order to avoid the irrelevant calculation minutia when adopting an independent matrix with a large rank and only to highlight inner transformation, herein, a simple multiple linear regression model $y = X\hat{\beta} + \hat{\varepsilon}$ is adopted with the in-

dependent matrix $X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ and the dependent vector $y = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\hat{\beta}$ symbolizes the regression coefficient estimator vector.

In this multiple linear regression model, the matrix X 's singular value decomposition (SVD) can be demonstrated as below:

$$X = U\theta V^T = \begin{bmatrix} 0.324 & 0.854 & 0.420 \\ 0.549 & 0.186 & -0.820 \\ 0.774 & -0.490 & 0.410 \end{bmatrix} \begin{bmatrix} 4.07 & 0 \\ 0 & 0.6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.403 & 0.915 \\ 0.915 & -0.403 \end{bmatrix}^T \quad (E.1)$$

U and V are respectively the left and right singular vector matrix of X , and θ is eigen value matrix of X [6] [7] [17].

Because X is a matrix with full rank column, $\hat{\beta}$ can be demonstrated as $\hat{\beta} = X^+y$ [13] [16], X^+ signifies the Moore-Penrose generalized inverse of matrix X . And $\hat{\beta}$ can also be demonstrated as below when X is substituted by its SVD form presented in equation (E.1),

$$\hat{\beta} = X^+y = (U\theta V^T)^+ y = (V^T)^{-1} \theta^+ U^{-1}y \quad (E.2)$$

The Equation (E.2) can be regarded as a process that the vector y is transformed 3 times from right to left multiplied by $(V^T)^{-1} \theta^+ U^{-1}$.

Transformation 1

$$y_1 = U^{-1}y = \begin{bmatrix} 0.324 & 0.854 & 0.420 \\ 0.549 & 0.186 & -0.820 \\ 0.774 & -0.490 & 0.410 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1.86 \\ -0.12 \\ 1.24 \end{bmatrix} \quad (E.3)$$

$$U\text{'s columns are demonstrated as } u_1 = \begin{bmatrix} 0.324 \\ 0.549 \\ 0.774 \end{bmatrix}, u_2 = \begin{bmatrix} 0.854 \\ 0.186 \\ -0.490 \end{bmatrix} \text{ and}$$

$$u_3 = \begin{bmatrix} 0.420 \\ -0.820 \\ 0.410 \end{bmatrix}.$$

Let the coordinate axes of a 3-dimension original coordinate system be symbolized by $a_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $a_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. The quintessence of the equation

(E.3) is that multiplied by the matrix U^{-1} , vector $y = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ in the coordinate

system constructed by a_1, a_2 and a_3 , can be transformed into $y_1 = \begin{bmatrix} 1.86 \\ -0.12 \\ 1.24 \end{bmatrix}$

which is located in a coordinate system constructed by u_1, u_2 and u_3 , or left space of X . In fact, y and y_1 are at the same spatial location. However, y is in the coordinate system constructed by a_1, a_2 and a_3 , meanwhile y_1 is presented by the coordinate system constructed by u_1, u_2 and u_3 , in another word, y_1 is in the left space of X . These locations can be demonstrated as below in **Figure 1**.

Transformation 2

$$y_2 = \theta^+ U^{-1} y = \theta^+ y_1 = \begin{bmatrix} 4.07 & 0 \\ 0 & 0.6 \\ 0 & 0 \end{bmatrix}^+ \begin{bmatrix} 1.86 \\ -0.12 \\ 1.24 \end{bmatrix} = \begin{bmatrix} 0.46 \\ -0.20 \\ 0 \end{bmatrix} \tag{E.4}$$

During this transformation, multiplied by θ 's Moore–Penrose inverse θ^+ , the vector y_1 in u_1, u_2 and u_3 coordinate system is transformed into the vector y_2 which is in the same coordinate system as y_1 . Such a transformation can be called “Vector Stretching”. During this transformation, the coordinate value of y_1 in u_1 axis diminishes, meanwhile, the coordinate value of y_1 in u_2 axis gets enlarged. The coordinate value of y_1 in u_3 axis vanishes. The “Vector Stretching” of y_1 into y_2 can be demonstrated as below in **Figure 2**.

The vanishing coordinate value of y_1 in u_3 axis can signify the degree of freedom of X from a geometric view.

Transformation 3

$$\hat{\beta} = (V^T)^{-1} \theta^+ U^{-1} y = (V^T)^{-1} y_2 = \left(\begin{bmatrix} 0.403 & 0.915 \\ 0.915 & -0.403 \end{bmatrix}^T \right)^{-1} \begin{bmatrix} 0.46 \\ -0.20 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} \tag{E.5}$$

X 's right singular vectors can be denoted as $v_1^T = \begin{bmatrix} 0.403 \\ 0.915 \end{bmatrix}$ and $v_2^T = \begin{bmatrix} 0.915 \\ -0.403 \end{bmatrix}$. v_1^T and v_2^T vectors are located in a coordinate system constructed by u_1 and u_2 vectors [16].

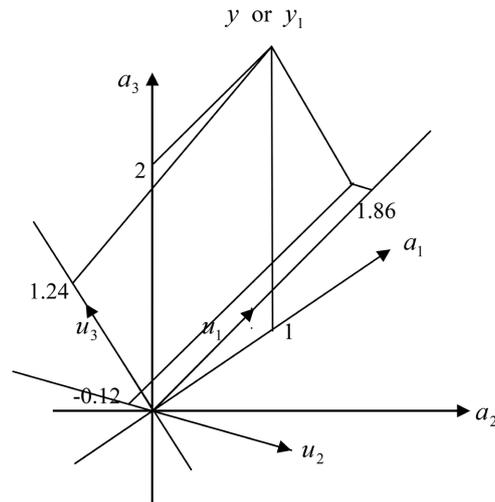


Figure 1. The dependent variable vector y transformed into y_1 by landing on to the left space of matrix X .

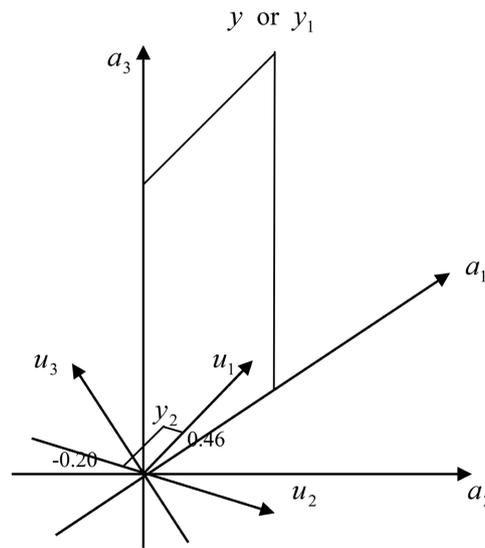


Figure 2. Vector y_1 transformed into y_2 through “Vector Stretching”.

The quintessence of Equation (E.5) is that the vector y_2 in the coordinate system is constructed by u_1 and u_2 can be embodied by the vector $\hat{\beta}$ in the coordinate system v_1^T and v_2^T . This transformation is demonstrated as below in **Figure 3**.

Though the vector $\hat{\beta}$ has the same spatial position as y_2 in **Figure 3**, these two vectors are situated in two different coordinate systems constructed respectively by v_1^T, v_2^T (the right space of X) and u_1, u_2 (the left space of X). In another word, the distinction between the vector $\hat{\beta}$ and y_2 is that $\hat{\beta}$ is demonstrated by the coordinate system of v_1^T and v_2^T , meanwhile, y_2 is demonstrated by u_1 and u_2 . As demonstrated by the aforementioned 3 steps of transformation, the dependent variable y vector is inner-transformed into parameter $\hat{\beta}$ vector.

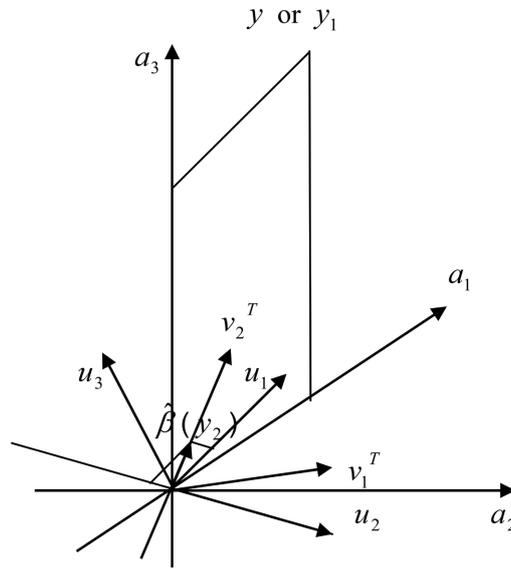


Figure 3. Vector y_2 transformed into $\hat{\beta}$ by landing on to the right space of matrix X .

3. y Can Be Transformed into \hat{y}

The independent variable X 's left singular vector matrix U [6] [7] [17]) can be blocked into two sub-matrices U_1 and U_2 as below.

$$U = \begin{bmatrix} 0.324 & 0.854 & 0.420 \\ 0.549 & 0.186 & -0.820 \\ 0.774 & -0.490 & 0.410 \end{bmatrix} = [U_1 \mid U_2] = \begin{bmatrix} 0.324 & 0.854 & \mid & 0.420 \\ 0.549 & 0.186 & \mid & -0.820 \\ 0.774 & -0.490 & \mid & 0.410 \end{bmatrix}$$

Let \hat{y} symbolize the estimator of the dependent variable y . For that U_1 is the left space of X , \hat{y} can be demonstrated as $\hat{y} = X\hat{\beta} = U_1U_1^T y$ [16], based on which the following result can be derived.

$$\hat{y} = U_1U_1^T y = \begin{bmatrix} 0.324 & 0.854 \\ 0.549 & 0.186 \\ 0.774 & -0.490 \end{bmatrix} \begin{bmatrix} 0.324 & 0.854 \\ 0.549 & 0.186 \\ 0.774 & -0.490 \end{bmatrix}^T \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \\ 1.5 \end{bmatrix}$$

Because U_1 is an orthonormal matrix, U_1 's Moore-Penrose inverse U_1^+ is equal with U_1^T [12]. So, \hat{y} can be demonstrated as below:

$$\hat{y} = U_1U_1^+ y = \begin{bmatrix} 0.324 & 0.854 \\ 0.549 & 0.186 \\ 0.774 & -0.490 \end{bmatrix} \begin{bmatrix} 0.324 & 0.854 \\ 0.549 & 0.186 \\ 0.774 & -0.490 \end{bmatrix}^+ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \\ 1.5 \end{bmatrix} \tag{E.6}$$

Based on E.6, y can be transformed into \hat{y} by two steps as below.

Transformation 4

$$y^* = U_1^+ y = \begin{bmatrix} 0.324 & 0.854 \\ 0.549 & 0.186 \\ 0.774 & -0.490 \end{bmatrix}^+ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1.86 \\ -0.12 \end{bmatrix} \tag{E.7}$$

In Transformation 4, y transforms into y^* by being projected into U_1 , the

left space of X . Another word, y^* is the projection of y in X .

Transformation 5

$$\hat{y} = U_1 U_1^+ y = U_1 y^* = \begin{bmatrix} 0.324 & 0.854 \\ 0.549 & 0.186 \\ 0.774 & -0.490 \end{bmatrix} \begin{bmatrix} 1.86 \\ -0.12 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \\ 1.5 \end{bmatrix} \tag{E.8}$$

In Transformation 5, multiplied by U_1 , y^* transforms into \hat{y} and is presented in the coordinate system constructed by a_1, a_2 and a_3 . Their spatial locations are demonstrated in **Figure 4**.

In **Figure 4**, it can be found that the error estimator $\hat{\varepsilon}$ is located in the same direction with u_3 which is the left null space of matrix X , and can be expressed as below:

$$\hat{\varepsilon} = \begin{bmatrix} 0.5 \\ -1 \\ 0.5 \end{bmatrix} \cong 1.22 \begin{bmatrix} 0.42 \\ -0.82 \\ 0.41 \end{bmatrix} = 1.22u_3.$$

\hat{y} is the projection of y into the left space of X , but presented in the original coordinate system constructed by a_1, a_2 and a_3 . $\hat{\varepsilon}$ is located in the left null space of matrix X and perpendicular with \hat{y} . This perpendicular result can be geometrically demonstrated in **Figure 4** as well as in the following multiplication.

$$\hat{y}^T \hat{\varepsilon} = [0.5 \quad 1 \quad 1.5] \begin{bmatrix} 0.5 \\ -1 \\ 0.5 \end{bmatrix} = 0.$$

4. Finding and Conclusion

By applying SVD and Moore-Penrose generalized inverse of the independent variable X in a multiple linear regression model, the dependent variable y can be transformed into the regression coefficient estimator vector $\hat{\beta}$ and its own estimator \hat{y} . This process presents a new geometric perspective to study the relationship between $X, y, \hat{\beta}$ and \hat{y} , through the inner-transformation algorithm.

As demonstrated from **Figures 1-3**, y transforms into $\hat{\beta}$ by transforming from the original coordinate system of a_1, a_2 and a_3 into the right space of X constructed by v_1^T and v_2^T . Through this process, y transforms into y_1 by transferring into the coordinate system of u_1, u_2 and u_3 , that is presented by Transformation 1 (E.3) and demonstrated in **Figure 1**.

Multiplied by θ^+ , the Moore-Penrose inverse of θ , y_1 stretches into y_2 , that is presented by Rotation 2 (E.4) and demonstrated in **Figure 2**. In this transformation, the coordinate value of y_1 in u_3 axis vanished, which also signifies the degree of freedom of X from a geometric view.

Multiplied by $(V^T)^{-1}$, the inverse of the right singular value matrix of X , y_2 transforms into $\hat{\beta}$, that is presented by Rotation 3 (E.5) and demonstrated in **Figure 3**.

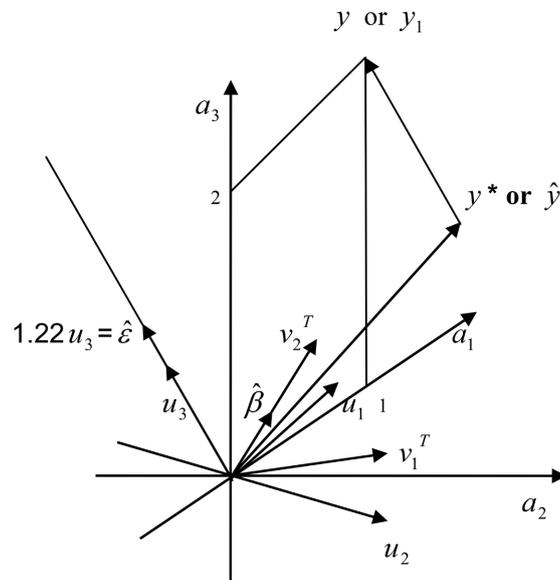


Figure 4. The dependent variable estimator \hat{y} landing on to the left space of X , meanwhile, the error estimator $\hat{\varepsilon}$ landing on to the left null space of X .

As demonstrated in **Figure 4**, y transforms into y^* by projecting into U_1 , the left space of X . And then, multiplied by U_1 , y^* transforms into \hat{y} and returns back into the original coordinate system constructed by a_1, a_2 and a_3 . \hat{y} is the projection of y into the left space of X , and perpendicular with error estimator $\hat{\varepsilon}$ which is located in the left null space of X .

With the aid of the algorithms like matrix decomposition and Moore-Penrose generalized matrix inverse, the dependent variable y of a multiple linear regression model can be inner-transformed into the regression coefficient estimator $\hat{\beta}$ and its own estimator \hat{y} . This process is a new approach to illustrate the inner-transformation between variables from a geometric view as well as presenting the spatial locations of the variables. To date, there is no study to explore the relationship between the variables of the multiple linear regression model from the view of geometric transformation. This study fills such a gap and provides a new perspective for studying multiple linear regression.

The limitation of this work is that a simple example of the multiple linear model is adopted to present such intricate inner transformation. But, it can demonstrate more inference art of inner transformation if more complex examples of multiple linear regression model are adopted in future studies.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

- [1] Mandel, J. (1982) Use of the Singular Value Decomposition in Regression Analysis.

- The American Statistician*, **36**, 15-24.
<https://doi.org/10.1080/00031305.1982.10482771>
- [2] Mishra, A., Dey, D.K. and Chen, K. (2017) Sequential Co-Sparse Factor Regression. *Journal of Computational and Graphical Statistics*, **26**, 814-826.
<https://doi.org/10.1080/10618600.2017.1340891>
- [3] Nelder, J.A. (1985) An Alternative Interpretation of Singular Value Decomposition in Regression. *The American Statistician*, **39**, 63-65.
<https://doi.org/10.2307/2683911>
- [4] De Schutter, B. and De Moor, B. (1998) The QR Decomposition and the Singular Value Decomposition in the Symmetrized Max-Plus Algebra. *SIAM Journal on Matrix Analysis and Applications*, **19**, 378-406.
<https://doi.org/10.1137/S0895479896304782>
- [5] Kalman, D. (1996) A Singularly Valuable Decomposition: SVD of a Matrix. *The College Mathematics Journal*, **27**, 2-23.
<https://doi.org/10.1080/07468342.1996.11973744>
- [6] Stewart G.W. (1993) On the Early History of the Singular Value Decomposition. *SIAM Review*, **35**, 551-566. <https://doi.org/10.1137/1035134>
- [7] Stewart, G.W. (1998) Matrix Algorithms: Volume 1: Basic Decompositions. Society for Industrial and Applied Mathematics, Philadelphia.
<https://doi.org/10.1137/1.9781611971408>
- [8] Ben-Israel, A. and Charnes, A. (1963) Contributions to the Theory of Generalized Inverses. *Journal of the Society for Industrial and Applied Mathematics*, **11**, 667-699.
<https://doi.org/10.1137/0111051>
- [9] Campbell, S.L. and Meyer, C.D. (1991) Generalized Inverse of Linear Transformations. Dover Publications, New York.
- [10] Fill, J.A. and Fishkind, D.E. (2000) The Moore—Penrose Generalized Inverse for Sums of Matrices. *SIAM Journal on Matrix Analysis and Applications*, **21**, 629-635.
<https://doi.org/10.1137/S0895479897329692>
- [11] Penrose, R. (1955) A Generalized Inverse for Matrices. *Mathematical Proceedings of the Cambridge Philosophical Society*, **51**, 406-413.
<https://doi.org/10.1017/S0305004100030401>
- [12] Wang, G.R., Wei, Y.M. and Qiao, S. (2003) Generalized Inverse: Theory and Computations. Science Press, Beijing.
- [13] Tian, Y. and Zhang, J. (2011) Some Equalities for Estimations of Partial Coefficients under a General Linear Regression Model. *Statistical Papers*, **52**, 911-920.
<https://doi.org/10.1007/s00362-009-0298-5>
- [14] Xu, W. and Tang, X. (2009) Ridge Estimate on Moore-Penrose Inverse Matrix of Generalized Linear Regression Model. *Journal of Northeast Forestry University*, **37**, 108-119.
- [15] Rao, C.R. and Mitra, S.K. (1972) Generalized Inverse of a Matrix and Its Applications. In: Le Cam, L.M., Neyman, J. and Scott, E.L., Eds., *Proceedings of the Sixth Berkeley Symposium on Statistics and Probability, Berkeley and Los Angeles, Vol. 1: Theory of Statistics*, University of California Press, 355-372.
- [16] Li, Z.Y. (2006) Matris Ayırımı. Yüksek Lisans Tezi. Master Degree Thesis, İstanbul Üniversitesi, İstanbul.
- [17] Piziak, R. and Odell, P.L. (1999) Full Rank Factorization of Matrix. *Mathematics Magazine*, **72**, 193-201. <https://doi.org/10.1080/0025570X.1999.11996730>